

Hierarchy-Dependent Suppression of Kozai-Lidov Oscillations by General Relativity

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ABSTRACT

We study the competition between Kozai–Lidov (KL) oscillations and general relativistic (GR) apsidal precession in hierarchical triple systems containing an inner double white dwarf (DWD) binary and a tertiary companion. The tertiary induces secular oscillations that exchange angular momentum with the inner binary, driving large eccentricities at high mutual inclination, while GR precession suppresses this growth. The transition between KL-dominated and GR-suppressed regimes is commonly characterized by the timescale ratio $\chi = t_{\text{KL}}/t_{\text{GR}} \sim 1$, though the universality of this criterion remains uncertain.

We explore this transition using a phenomenological model that recasts GR suppression as an effective angular momentum floor, limiting the maximum eccentricity attainable during KL cycles. We apply this model across a grid of hierarchical DWD triples spanning a range of inclinations and hierarchies, and validate key trends using direct N -body integrations with **REBOUND** and relativistic corrections via **REBOUNDx**.

We find that the KL–GR transition is not described by a universal $\chi = 1$ boundary. Instead, the transition location depends systematically on hierarchy, while the width varies only weakly. GR suppression therefore operates as a gradual, architecture-dependent process rather than a sharp cutoff, with implications for merger predictions in compact object triples.

1. INTRODUCTION

Hierarchical triple systems provide a natural environment for long-term secular dynamical evolution. In such systems, a distant tertiary companion can induce Kozai–Lidov (KL) oscillations in the inner binary, leading to periodic exchanges between inclination and eccentricity (Naoz 2016). When the mutual inclination lies within the approximate range $40^\circ \lesssim i \lesssim 140^\circ$, these oscillations can drive the inner binary to extreme eccentricities without significant energy exchange, instead redistributing angular momentum within the system.

This mechanism is particularly important for compact object binaries, including double white dwarf (DWD) systems, where high eccentricities can dramatically shorten merger timescales through enhanced gravitational wave emission. However, additional physical effects can modify or suppress KL evolution. In particular, general relativistic (GR) apsidal precession introduces an additional source of periapsis advance that competes with the KL mechanism (Blaes et al. 2002; Fabrycky & Tremaine 2007; Liu et al. 2014). When GR precession becomes sufficiently strong, it can quench the eccentricity growth driven by KL oscillations.

A common framework for understanding this competition compares the KL and GR precession timescales through the ratio $\chi = t_{\text{KL}}/t_{\text{GR}}$. In many studies, the transition between KL-dominated and GR-suppressed

regimes is approximated as occurring near $\chi \sim 1$, and this criterion is often used as a sharp boundary when estimating merger rates or identifying dynamically active systems.

However, the efficiency of KL oscillations and their susceptibility to GR suppression depend on additional parameters, including hierarchy and inclination (Liu et al. 2014; Martin & Triaud 2015). Whether the commonly used $\chi \sim 1$ criterion provides a universal description of the transition therefore remains uncertain.

In this work, we revisit the KL–GR transition in hierarchical triples containing DWD inner binaries. We develop a phenomenological analytic model that captures GR suppression as an effective angular momentum floor, limiting the maximum eccentricity attainable during KL cycles.

We apply this model across a broad parameter space, varying hierarchy and inclination, and examine how the transition behavior changes across different system architectures. To validate our analytic framework, we perform direct N -body integrations using **REBOUND** with relativistic corrections implemented through **REBOUNDx**.

2. ANALYTICAL FRAMEWORK

We model the competition between KL oscillations and GR precession using a phenomenological framework designed to capture the transition between eccentricity

excitation and suppression across a broad population of hierarchical triples.

2.1. System Setup and Control Parameters

We consider hierarchical triple systems consisting of an inner binary with semi-major axis a_1 and an outer tertiary with semi-major axis a_2 . We define the hierarchy parameter as

$$\eta = \frac{a_2}{a_1}. \quad (1)$$

The competition between KL oscillations and GR precession is characterized by the dimensionless control parameter

$$\chi = \frac{t_{\text{KL}}}{t_{\text{GR}}}, \quad (2)$$

where t_{KL} is the quadrupole-order KL timescale (Antognini 2015) and t_{GR} is the relativistic precession timescale of the inner binary.

The parameter χ compares the characteristic secular forcing timescale to the relativistic precession timescale. We treat it as an organizing coordinate and test whether the transition occurs at a fixed value across different hierarchies.

2.2. KL-Only Maximum Eccentricity

Before introducing GR suppression, we first consider the eccentricity evolution expected from KL oscillations alone. In the quadrupole limit, KL cycles periodically exchange inclination and eccentricity while approximately conserving the secular energy and angular momentum of the system. As a result, the maximum eccentricity reached during a KL cycle depends primarily on the initial mutual inclination between the inner and outer orbits.

For initially near-circular inner binaries, the standard quadrupole-order result gives

$$e_{\text{max,KL}}^2 = 1 - \frac{5}{3} \cos^2 i_0, \quad (3)$$

where i_0 is the initial mutual inclination.

This expression captures the well-known inclination dependence of KL oscillations: systems near 90° can reach extremely large eccentricities, while systems outside the KL-active inclination window experience little eccentricity excitation. In terms of the dimensionless angular momentum,

$$j \equiv \sqrt{1 - e^2}, \quad (4)$$

larger KL eccentricities correspond to progressively smaller accessible values of j .

This KL-only eccentricity scale provides the reference point for our GR suppression model. The key idea developed in the following section is that GR precession

introduces an effective lower limit on the angular momentum that the inner binary can reach. The competition between the KL-accessible angular momentum scale and this GR-induced floor then determines how strongly eccentricity growth is suppressed.

2.3. GR Suppression as an Angular Momentum Floor

GR precession suppresses KL-driven eccentricity growth by introducing rapid periapsis advance in the inner binary as the orbit approaches extreme eccentricity. In the KL mechanism, secular torques exchange angular momentum between the inner and outer orbits, allowing the inner binary to reach very small angular momentum and therefore very large eccentricity. As GR precession strengthens, the periapsis advances too rapidly for the secular KL torque to maintain coherent eccentricity growth.

Rather than treating this suppression as a sharp dynamical boundary, we model it phenomenologically as an effective floor on the dimensionless angular momentum j . In this picture, KL oscillations drive the system toward progressively smaller angular momentum, while GR precession prevents the orbit from accessing arbitrarily low- j states.

The motivation for this approach follows from the standard comparison between KL forcing and relativistic apsidal precession rates (Blaes et al. 2002; Fabrycky & Tremaine 2007; Liu et al. 2014). At high eccentricity,

$$\dot{\omega}_{\text{GR}} \propto \frac{1}{t_{\text{GR}} j^2}, \quad (5)$$

so the GR precession becomes increasingly rapid as $j \rightarrow 0$. By contrast, the characteristic KL driving rate is approximately

$$\dot{\omega}_{\text{KL}} \sim \frac{1}{t_{\text{KL}}}. \quad (6)$$

Suppression occurs once the GR precession becomes fast enough to compete with the KL torque,

$$\frac{1}{t_{\text{GR}} j^2} \sim \frac{1}{t_{\text{KL}}}. \quad (7)$$

Defining the control parameter

$$\chi \equiv \frac{t_{\text{KL}}}{t_{\text{GR}}}, \quad (8)$$

this balance gives the scaling

$$j_{\text{GR}}^2 \propto \chi. \quad (9)$$

We therefore interpret GR suppression as imposing a minimum accessible angular momentum scale,

$$j_{\text{eff}}^2 = \max(j_{\text{KL}}^2, j_{\text{GR}}^2), \quad (10)$$

where

$$j_{\text{KL}}^2 = 1 - e_{\text{max,KL}}^2 \quad (11)$$

is the angular momentum associated with the KL-only maximum eccentricity.

The corresponding GR-suppressed eccentricity is then

$$e_{\text{max,KL+GR}} = \sqrt{1 - j_{\text{eff}}^2}. \quad (12)$$

This framework naturally produces a gradual transition between KL-dominated and GR-suppressed behavior. When KL oscillations are still able to reach angular momentum scales below the GR floor, large eccentricity growth persists. As the GR floor rises, the accessible eccentricity range becomes progressively restricted, reducing the maximum eccentricity attainable by the inner binary.

This explains why a simple $\chi \sim 1$ criterion is insufficient. Although χ organizes the transition globally, suppression depends on how the GR-induced floor intersects the KL-accessible eccentricity range. Systems with different hierarchy can therefore exhibit similar transition shapes while reaching suppression at different characteristic values of χ .

Because compact-object merger times depend sensitively on the maximum eccentricity, this transition directly influences merger outcomes.

2.4. Merger Timescale and Population Diagnostics

To connect eccentricity excitation to astrophysical outcomes, we compute merger times using the Peters formalism (Peters 1964). For each system, we evaluate both KL-only and KL+GR eccentricities across a grid of mutual inclinations and compute the corresponding merger timescales.

We define merger windows based on whether KL excitation reduces the inspiral time below a fiducial Hubble timescale. Rather than applying a perfectly sharp cut-off at this boundary, we use a smooth weighting based on the proximity of the merger time to the Hubble timescale, allowing us to define effective window widths and suppression fractions continuously.

Comparing the KL-only and KL+GR cases provides a direct diagnostic of how GR reshapes the merger-producing region of parameter space.

2.5. Population Sampling and Assumptions

We construct the population by initially sampling a_1 and η uniformly in logarithmic space and enforcing dynamical stability using a Mardling–Aarseth criterion (Mardling & Aarseth 2001). The outer semi-major axis is set by $a_2 = \eta a_1$, and systems exceeding a maximum outer separation are excluded.

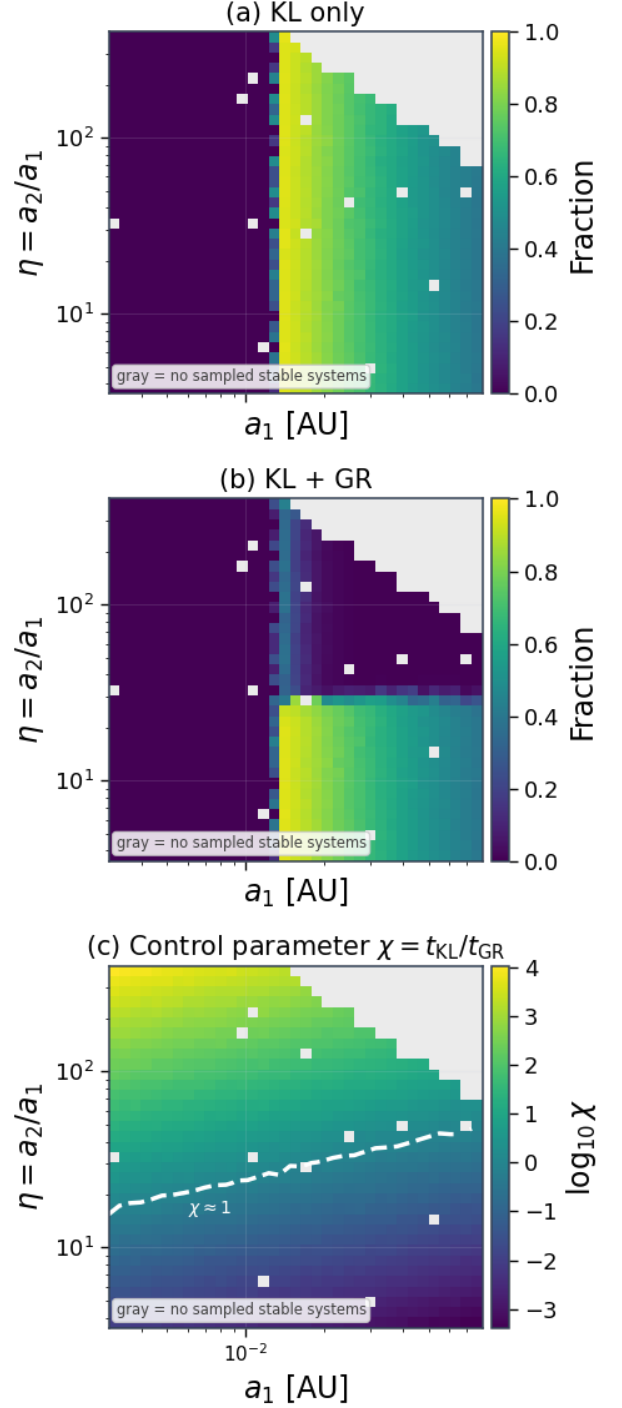


Figure 1. Population-level GR suppression in the analytic KL merger model. (a) Effective merger fraction for KL-only evolution. (b) Effective merger fraction including GR precession. (c) Control parameter $\chi = t_{\text{KL}}/t_{\text{GR}}$ across the same parameter space. The white dashed line denotes the nominal $\chi = 1$ boundary, while comparison with panel (b) shows that the transition to strong GR suppression occurs gradually rather than at a single universal threshold.

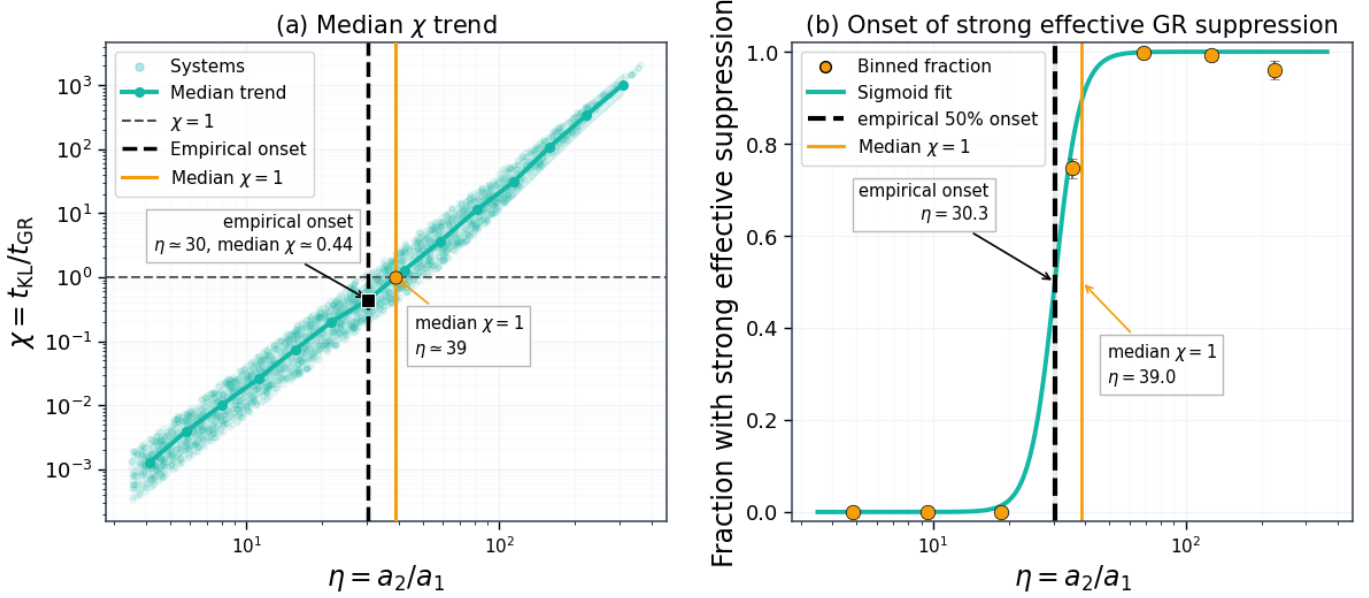


Figure 2. Hierarchy dependence of the KL-GR transition in the analytic population model. (a) Median value of the control parameter $\chi = t_{\text{KL}}/t_{\text{GR}}$ as a function of hierarchy $\eta = a_2/a_1$. The black dashed line marks the empirically determined suppression onset from panel (b), while the gold line indicates where the median trend reaches $\chi = 1$. The offset between these lines demonstrates that the effective suppression transition departs systematically from the canonical $\chi \sim 1$ expectation. (b) Fraction of systems exhibiting strong effective suppression as a function of hierarchy. The black dashed line marks the empirical 50% suppression onset obtained from the sigmoid fit, while the gold line again denotes the hierarchy where the median trend in panel (a) reaches $\chi = 1$.

2.6. Transition Extraction and Width Measurements

To quantify the KL-GR transition, we analyze the maximum eccentricity as a function of χ at fixed inclination and hierarchy. For each transition curve, we define the normalized eccentricity

$$y_{\text{norm}} = \frac{e_{\text{max}} - e_{\text{low}}}{e_{\text{high}} - e_{\text{low}}}, \quad (13)$$

where e_{high} and e_{low} are the asymptotic high- and low-eccentricity limits.

We define the characteristic transition location χ_0 using the midpoint condition

$$y_{\text{norm}} = 0.5. \quad (14)$$

To isolate the transition shape independently of its location, we define the width using the interval between the 20% and 80% suppression levels,

$$w_{20-80} = \log_{10} \chi(y_{\text{norm}} = 0.2) - \log_{10} \chi(y_{\text{norm}} = 0.8). \quad (15)$$

This procedure separates shifts in the transition location from changes in the transition morphology. Re-centered curves in χ/χ_0 then allow direct comparison of the intrinsic transition structure across different hierarchies.

Because the analytic model is inexpensive to evaluate, it enables efficient exploration of broad parameter space before direct integrations.

3. POPULATION-LEVEL KL-GR SUPPRESSION

We first use the framework described in Section 2 to examine GR suppression at the population level by evaluating the effective merger fraction across a grid of hierarchical systems. For each system, we compute the fraction of inclination space that produces mergers within a Hubble timescale, both with and without GR.

Figure 1 shows these results across parameter space. In the KL-only case, efficient mergers occur across a broad range of hierarchies, with the dominant structure primarily following the inner binary separation.

Including GR substantially alters this picture. A large fraction of the previously active parameter space becomes strongly suppressed, particularly at high hierarchy where the KL torque operates more weakly and slowly. As a result, GR precession more easily disrupts the secular eccentricity growth required for mergers.

The transition between KL-dominated and GR-suppressed behavior remains relatively sharp overall, but still retains a finite region of intermediate partial suppression near the transition boundary.

4. SUPPRESSION VS CONTROL PARAMETER χ

To better understand the structure of the suppression, Figure 1c shows the distribution of the control parameter $\chi = t_{\text{KL}}/t_{\text{GR}}$ across the same parameter space.

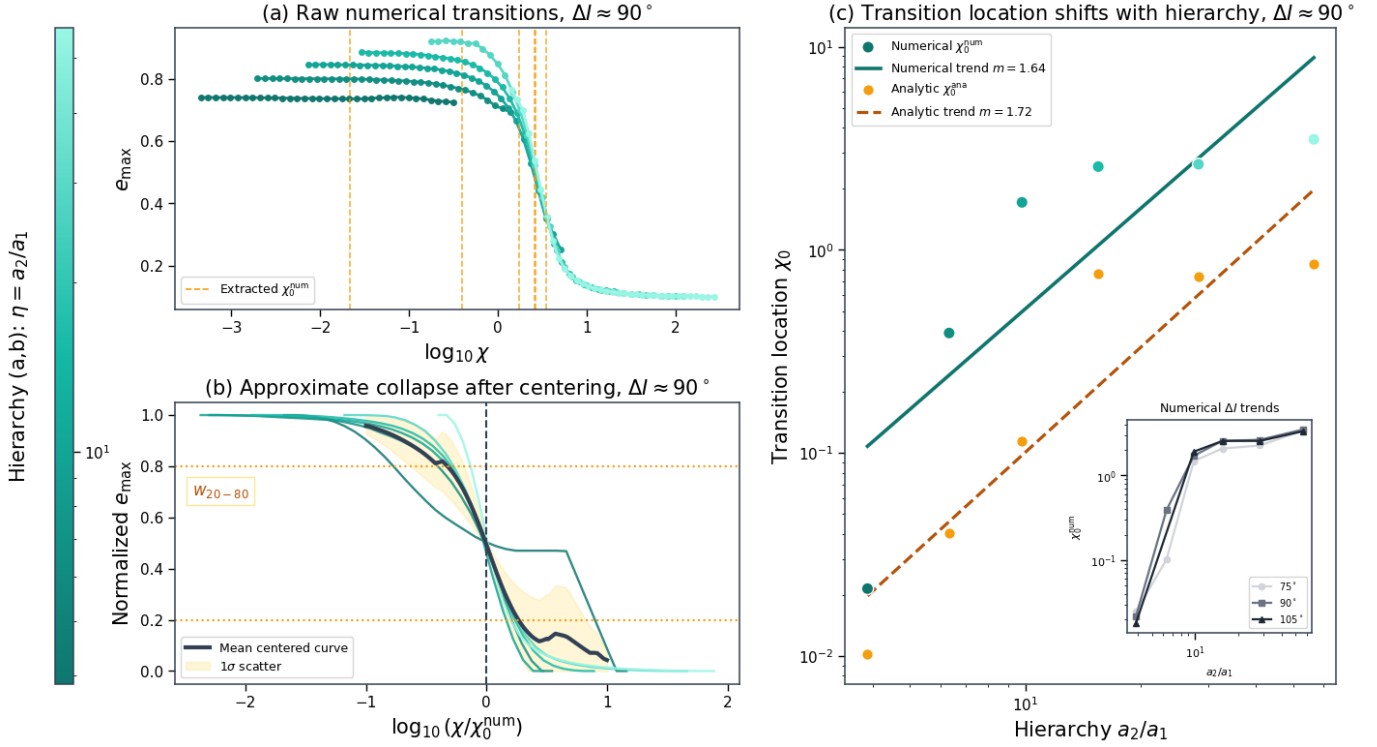


Figure 3. Numerical validation of the KL-GR transition. (a) Raw eccentricity transitions showing hierarchy-dependent transition locations. (b) Approximate collapse after recentering by χ_0^{num} , indicating a broadly shared transition profile with modest residual hierarchy dependence. (c) Transition location as a function of hierarchy, comparing numerical and analytic trends. The analytic model captures the overall behavior but slightly overestimates the scaling with hierarchy.

Regions with large χ generally correspond to stronger GR suppression in Figure 1b, indicating that the relative strength of the KL and GR timescales plays a central role in organizing the transition.

However, the suppression does not appear to follow a single sharp universal boundary such as $\chi \sim 1$. Instead, the transition develops gradually across parameter space, suggesting that the KL-GR competition depends not only on the timescale ratio itself, but also on the underlying structure of the triple system.

5. HIERARCHY-DEPENDENT TRANSITION SCALE

While the previous section established that $\chi = t_{\text{KL}}/t_{\text{GR}}$ acts as a useful global organizing parameter, the population-level results also suggest that the transition is not characterized by a single universal threshold.

Figure 2 shows that the effective KL-GR transition departs substantially from the canonical $\chi \sim 1$ expectation, with the median transition scale shifting toward larger values of χ at higher hierarchy. More hierarchical systems therefore require stronger GR precession relative to KL forcing before eccentricity growth becomes substantially suppressed.

However, both the observed offset and any apparent morphological broadening arise within a population that

mixes systems across a range of mutual inclinations. As a result, no strong conclusions about the intrinsic transition structure can be drawn from the population-level trends alone. These discrepancies therefore motivate the fixed-inclination analytic and numerical analysis that follows.

Using the extraction procedure described in Section 2.6, we measure the characteristic transition scale χ_0 and transition width w_{20-80} for individual hierarchy bins and inclination slices. To quantify the systematic shift of the transition with hierarchy, we fit the extracted χ_0 values using the phenomenological relation

$$\chi_0 \propto \left(\frac{a_2}{a_1} \right)^m, \quad (16)$$

where m characterizes how rapidly the transition moves across the sampled hierarchy range. The resulting analytic trends are shown in Figure 3c. The purpose of this fit is not to establish a precise empirical scaling relation, but rather to emphasize the systematic hierarchy-dependent displacement of the transition.

6. NUMERICAL VALIDATION OF THE TRANSITION

To validate the analytic model we perform direct N -body integrations using REBOUND (Rein & Liu 2012) with

relativistic corrections implemented through `REBOUNDx` (Tamayo et al. 2020). We focus on fixed inclination slices to isolate the intrinsic structure of the transition.

Figure 3a,b shows the fixed-inclination numerical results, while Figure 3c compares the numerical and analytic hierarchy scaling at a mutual inclination of 90° . The raw eccentricity curves exhibit a clear transition from KL-driven excitation to GR suppression, with the transition location shifting systematically with hierarchy.

After recentering by χ_0 , the transition curves collapse onto a similar shape, indicating that hierarchy primarily shifts the transition location while only weakly affecting the transition width.

We also find that the numerical scaling of χ_0 with hierarchy is shallower than predicted analytically, although the qualitative trend is preserved. This is expected given the simplified nature of the analytic model, which assumes a closed-form KL eccentricity and a single effective GR angular momentum floor, whereas the direct integrations capture the coupled orbital evolution more fully. Residual deviations at lower hierarchy likely reflect stronger non-secular effects and increased dynamical complexity, causing the numerical transitions to collapse less cleanly onto a shared profile. Despite these differences, both approaches indicate that hierarchy primarily shifts the transition location while only weakly affecting the transition shape. At the highest sampled hierarchies, both the analytics and numerics also show evidence of partial flattening in the transition location, although the limited number of high- η bins prevents a strong constraint on this behavior.

7. DISCUSSION & CONCLUSIONS

We have investigated the transition between Kozai–Lidov excitation and general relativistic suppression in hierarchical triple systems using a combination of analytic modeling and numerical validation.

Our results show that the KL–GR transition is not characterized by a universal $\chi = 1$ boundary. Instead, the transition is gradual and depends systematically on system hierarchy, with more hierarchical systems requiring larger values of χ before strong suppression occurs. The transition width remains comparatively stable across hierarchy, indicating a broadly shared underlying transition morphology.

Although simplified, the analytic model reproduces the main features of the transition, including the hierarchy-dependent shift in χ_0 and the approximately invariant transition width. Fixed-inclination analytics and numerical integrations further show that much of the apparent broadening seen at the population level arises from mixing systems across different mutual inclinations rather than from large intrinsic changes in the transition structure itself.

Together, these results support a picture in which GR suppression acts as an angular momentum barrier rather than a simple on/off switch, progressively limiting access to extreme eccentricities.

These findings have important implications for merger predictions in compact-object triples, where merger times depend sensitively on the maximum eccentricity reached during KL evolution. A universal $\chi \sim 1$ criterion therefore provides only an approximate description of the merger-producing region, particularly near the transition boundary. Treating the transition as gradual and hierarchy-dependent provides a more physically accurate description of the secular dynamics governing compact triple evolution.

REFERENCES

- Antognini, J. M. O. 2015, *Monthly Notices of the Royal Astronomical Society*, 452, 3610, doi: [10.1093/mnras/stv1552](https://doi.org/10.1093/mnras/stv1552)
- Blaes, O., Lee, M. H., & Socrates, A. 2002, *The Astrophysical Journal*, 578, 775, doi: [10.1086/342655](https://doi.org/10.1086/342655)
- Fabrycky, D., & Tremaine, S. 2007, *The Astrophysical Journal*, 669, 1298–1315, doi: [10.1086/521702](https://doi.org/10.1086/521702)
- Liu, B., Muñoz, D. J., & Lai, D. 2014, *Monthly Notices of the Royal Astronomical Society*, 447, 747–764, doi: [10.1093/mnras/stu2396](https://doi.org/10.1093/mnras/stu2396)
- Mardling, R. A., & Aarseth, S. J. 2001, *Monthly Notices of the Royal Astronomical Society*, 321, 398, doi: [10.1046/j.1365-8711.2001.03974.x](https://doi.org/10.1046/j.1365-8711.2001.03974.x)
- Martin, D. V., & Triaud, A. H. M. J. 2015, *Monthly Notices of the Royal Astronomical Society: Letters*, 455, L46–L50, doi: [10.1093/mnrasl/slv139](https://doi.org/10.1093/mnrasl/slv139)
- Naoz, S. 2016, *Annual Review of Astronomy and Astrophysics*, 54, 441–489, doi: [10.1146/annurev-astro-081915-023315](https://doi.org/10.1146/annurev-astro-081915-023315)
- Peters, P. C. 1964, *Phys. Rev.*, 136, B1224, doi: [10.1103/PhysRev.136.B1224](https://doi.org/10.1103/PhysRev.136.B1224)
- Rein, H., & Liu, S.-F. 2012, *Astronomy and Astrophysics*, 537, A128, doi: [10.1051/0004-6361/201118085](https://doi.org/10.1051/0004-6361/201118085)
- Tamayo, D., Rein, H., Shi, P., & Hernandez, D. M. 2020, *Monthly Notices of the Royal Astronomical Society*, 491, 2885, doi: [10.1093/mnras/stz2870](https://doi.org/10.1093/mnras/stz2870)